(8) The Addition of Angular Momenta

① Tensor products and direct sums
[$\otimes$] [$\oplus$]

- They are to combine two Hilbert spaces (vector) into one.

a. tensor products.

- For a vector $\vec{v} \in V$ and the other $\vec{w} \in W$

$$\longrightarrow \quad \vec{v} \otimes \vec{w} \in V \otimes W.$$

ex. a system of two spin-$\frac{1}{2}$ particles.

the state ket of the system:

$$|\Psi\rangle = C_{\uparrow\uparrow} |\uparrow\rangle_1 \otimes |\uparrow\rangle_2 + C_{\uparrow\downarrow} |\uparrow\rangle_1 \otimes |\downarrow\rangle_2 + C_{\downarrow\uparrow} |\downarrow\rangle_1 \otimes |\uparrow\rangle_2 + C_{\downarrow\downarrow} |\downarrow\rangle_1 \otimes |\downarrow\rangle_2$$

- Addition of two operators that are in different H-spaces.

$$O_v \in V, \quad O_w \in W$$

$$\Longrightarrow \quad O_v + O_w \equiv O_v \otimes I_w + I_v \otimes O_w$$

- Operator (matrix) - ket (vector) multiplication

$$\rightarrow (S_1 + S_2)|\Psi\rangle = C_{\uparrow\uparrow}\left[(S_1|\uparrow\rangle_1) \otimes |\uparrow\rangle_2 + |\uparrow\rangle_1 \otimes (S_2 |\uparrow\rangle_2)\right] + \cdots$$

Rule ✦✦✦

$$(O_v \otimes I_w + I_v \otimes O_w) \cdot (|v\rangle \otimes |w\rangle)$$
$$= (O_v|v\rangle) \otimes (I_w|w\rangle) + (I_v|v\rangle) \otimes (O_w|w\rangle)$$

- base kets and dimension of the combined space.

$$V \; \{|v_i\rangle\} \;\otimes\; W \; \{|w_j\rangle\} \;\longrightarrow\; V\otimes W \; \{|v_i\rangle \otimes |w_j\rangle\}$$

$\dim[V]$ $\quad$ $\dim[W]$ $\quad$ $\dim[V\otimes W]$

$= \dim[V]\cdot\dim[W]$!

b. Direct sums.

$$V \; \{|v_i\rangle\} \;\oplus\; W \; \{|w_j\rangle\} \;\longrightarrow\; V\oplus W \; \{|v_i\rangle \cdots |w_j\rangle\}$$

$\dim[V]$ $\quad$ $\dim[W]$ $\quad$ $\dim[V\oplus W]$

$= \dim[V] + \dim[W]$

ex. matrix representation of $\mathcal{D}(R)$

$\Rightarrow \mathcal{D}^{(j)}_{m'm}(R)$

$$\begin{pmatrix} \boxed{j=1} & & \\ & \boxed{\quad j=2 \quad} & \\ & & \ddots \end{pmatrix} \Leftarrow V_{j=1} \oplus V_{j=2} \oplus \cdots$$

"Block - diagonal".

We have already seen the tensor product many times.

$\rightarrow \quad |x,y,z\rangle = |x\rangle \otimes |y\rangle \otimes |z\rangle$

$$H = \frac{\vec{P}^2}{2m} = \frac{1}{2m}\left(\left(\tilde{P}_x \otimes I_y \otimes I_z\right)^2 + \left(I_x \otimes \hat{P}_y \otimes I_z\right)^2 + \left(I_x \otimes I_y \otimes \tilde{P}_z\right)^2 \right)$$

C. notations.

1. Do NOT mix the index ordering.

ex. $\cancel{|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 + |\uparrow\rangle_2 \otimes |\downarrow\rangle_1}$ (X)

$|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 + |\downarrow\rangle_1 \otimes |\uparrow\rangle_2$ (O)

2. $\otimes$ (index) are often omitted for brevity.

ex. $|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 + |\downarrow\rangle_1 \otimes |\uparrow\rangle_2 \equiv |\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle$

: assuming that index ordering (1, 2) is fixed.

② Simple examples of Angular-Momentum addition

- base ket of a spin-$\frac{1}{2}$ particle in space.

$$|x;\uparrow\rangle \equiv |x\rangle \otimes |\uparrow\rangle, \quad |x;\downarrow\rangle \equiv |x\rangle \otimes |\downarrow\rangle$$

- Wave function :

$$\langle x;\uparrow|\alpha\rangle \equiv \psi_\uparrow(x), \quad \langle x;\downarrow|\alpha\rangle \equiv \psi_\downarrow(x).$$

or $\quad \Psi(x) \equiv \begin{pmatrix} \psi_\uparrow(x) \\ \psi_\downarrow(x) \end{pmatrix}$.

- L·S coupling (spin-orbit interaction)

$\vec{J} = \vec{L} + \vec{S} \qquad \left(\vec{L}\cdot\vec{S} = \vec{J}^2 - \vec{L}^2 - \vec{S}^2 \right)$

$\equiv \vec{L} \otimes I + I \otimes \vec{S}$ also. $\mathcal{D}(R) = \underset{(\vec{L})}{\mathcal{D}^{(orb)}(R)} \otimes \underset{(\vec{S})}{\mathcal{D}^{(spin)}(R)}$.

- Addition of $l=1$, $s=\frac{1}{2}$: $\vec{J}=\vec{L}+\vec{S}$

  - base vectors (kets)

$$|s, m_s\rangle : \quad |\tfrac{1}{2},\tfrac{1}{2}\rangle \doteq \begin{pmatrix}1\\0\end{pmatrix}, \quad |\tfrac{1}{2},-\tfrac{1}{2}\rangle \doteq \begin{pmatrix}0\\1\end{pmatrix}$$

$$|l, m_l\rangle : \quad |1,1\rangle \doteq \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad |1,0\rangle \doteq \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad |1,-1\rangle \doteq \begin{pmatrix}0\\0\\1\end{pmatrix}$$

  - Angular momentum operators

$$s=\tfrac{1}{2}: \quad S_+ \doteq \hbar\begin{pmatrix}0&1\\0&0\end{pmatrix}, \quad S_- \doteq \hbar\begin{pmatrix}0&0\\1&0\end{pmatrix}, \quad S_z \doteq \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$l=1: \quad L_+ \doteq \hbar\begin{pmatrix}0&\sqrt{2}&0\\0&0&\sqrt{2}\\0&0&0\end{pmatrix}, \quad L_- = \hbar\begin{pmatrix}0&0&0\\\sqrt{2}&0&0\\0&\sqrt{2}&0\end{pmatrix}, \quad L_z \doteq \hbar\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$$

$\downarrow$ in the form of the tensor product $(l=1)\otimes(s=\frac{1}{2})$

$$L_+\otimes I = \hbar \begin{array}{c} \begin{array}{cccccc} m_l=1 & & m_l=0 & & m_l=-1 & \\ & & m_s=\frac{1}{2} & m_s=-\frac{1}{2} & & \end{array} \\ \left(\begin{array}{cc|cc|cc} 0&0&\sqrt{2}&0&0&0\\ 0&0&0&\sqrt{2}&0&0\\ \hline 0&0&0&0&\sqrt{2}&0\\ 0&0&0&0&0&\sqrt{2}\\ \hline 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{array}\right) \end{array}, \quad L_-\otimes I = \hbar\left(\begin{array}{cc|cc|cc} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ \hline \sqrt{2}&0&0&0&0&0\\ 0&\sqrt{2}&0&0&0&0\\ \hline 0&0&\sqrt{2}&0&0&0\\ 0&0&0&\sqrt{2}&0&0 \end{array}\right)$$

$$I\otimes S_+ = \hbar\left(\begin{array}{cc|cc|cc} 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ \hline 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ \hline 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{array}\right), \quad I\otimes S_- = \hbar\left(\begin{array}{cc|cc|cc} 0&0&0&0&0&0\\ 1&0&0&0&0&0\\ \hline 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ \hline 0&0&0&0&0&0\\ 0&0&0&0&1&0 \end{array}\right)$$

$$L_z\otimes I = \hbar\left(\begin{array}{cc|cc|cc} 1&&&&&\\ &1&&&&\\ \hline &&0&&&\\ &&&0&&\\ \hline &&&&-1&\\ &&&&&-1 \end{array}\right), \quad I\otimes S_z = \frac{\hbar}{2}\left(\begin{array}{cc|cc|cc} 1&&&&&\\ &-1&&&&\\ \hline &&1&&&\\ &&&-1&&\\ \hline &&&&1&\\ &&&&&-1 \end{array}\right)$$

$$J_+ = L_+ \otimes I + I \otimes S_+ \quad , \quad J_- = L_- \otimes I + I \otimes S_-$$

$$= \hbar \left(\begin{array}{cc|cc|cc} 0 & 1 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad = \hbar \left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \hline \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 1 & 0 \end{array}\right)$$

$$J_z = L_z \otimes I + I \otimes S_z$$

$$= \hbar \left(\begin{array}{cc|cc|cc} \frac{3}{2} & & & & & \\ & \frac{1}{2} & & & & \\ \hline & & \frac{1}{2} & & & \\ & & & -\frac{1}{2} & & \\ \hline & & & & -\frac{1}{2} & \\ & & & & & -\frac{3}{2} \end{array}\right) \rightarrow$$

see if

$$[J_i , J_j] = i\hbar \varepsilon_{ijk} J_k$$

holds.

Yes!

eigenkets.

$$|j, m\rangle \rightarrow$$

It's obvious that

$$|\tfrac{3}{2}, \tfrac{3}{2}\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} m_\ell = 1 \;\; m_s = \frac{1}{2},\ m_s = -\frac{1}{2} \\ m_\ell = 0 \\ m_\ell = -1 \end{array} , \quad |\tfrac{3}{2}, -\tfrac{3}{2}\rangle \doteq \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}$$

* check if

$$\begin{cases} J_+ |\frac{3}{2}, \frac{3}{2}\rangle = 0 \\ J_- |\frac{3}{2}, -\frac{3}{2}\rangle = 0 \end{cases}$$

Lowering with J_- :

---- (*)

$$J_- |\tfrac{3}{2}, \tfrac{3}{2}\rangle = \hbar \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \hbar\sqrt{3}\, |\tfrac{3}{2}, \tfrac{1}{2}\rangle$$

From $\sqrt{(j+m)(j-m+1)}\,\hbar\, |j, m-1\rangle = J_- |j,m\rangle$

Raising with J_+ :

--- (**)

$$J_+ |\tfrac{3}{2}, -\tfrac{3}{2}\rangle = \hbar \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sqrt{2} \\ 1 \\ 0 \end{pmatrix} = \hbar\sqrt{3}\, |\tfrac{3}{2}, -\tfrac{1}{2}\rangle$$

From $\sqrt{(j-m)(j+m+1)}\,\hbar\, |j, m+1\rangle = J_+ |j,m\rangle$

also, by just using operators and kets.

$$J_-\left|\tfrac{3}{2},\tfrac{3}{2}\right\rangle = (L_-\otimes I + I\otimes S_-)\,(\overset{\ell\ \ m_\ell}{|1,1\rangle}\otimes\overset{s\ \ m_s}{\left|\tfrac12,\tfrac12\right\rangle})$$

$$= \hbar\sqrt{2}\,|1,0\rangle\otimes\left|\tfrac12,\tfrac12\right\rangle + \hbar\,|1,1\rangle\otimes\left|\tfrac12,-\tfrac12\right\rangle = \hbar\sqrt{3}\left|\tfrac32,\tfrac12\right\rangle$$

$$\Rightarrow \left|\tfrac32,\tfrac12\right\rangle = \sqrt{\tfrac23}\,|1,0\rangle\otimes\left|\tfrac12,\tfrac12\right\rangle + \sqrt{\tfrac13}\,|1,1\rangle\otimes\left|\tfrac12,-\tfrac12\right\rangle$$

$$\doteq \left(\begin{array}{c} 0 \\ 1 \\ \hdashline \sqrt{2} \\ 0 \\ \hdashline 0 \\ 0 \end{array}\right)$$

$$J_+\left|\tfrac32,-\tfrac32\right\rangle = (L_+\otimes I + I\otimes S_+)\,(|1,-1\rangle\otimes\left|\tfrac12,-\tfrac12\right\rangle)$$

$$= \hbar\,|1,-1\rangle\otimes\left|\tfrac12,\tfrac12\right\rangle + \hbar\sqrt{2}\,|1,0\rangle\otimes\left|\tfrac12,-\tfrac12\right\rangle = \hbar\sqrt{3}\left|\tfrac32,-\tfrac12\right\rangle$$

$$\Rightarrow \left|\tfrac32,-\tfrac12\right\rangle = \frac{1}{\sqrt3}\,|1,-1\rangle\otimes\left|\tfrac12,\tfrac12\right\rangle + \sqrt{\tfrac23}\,|1,0\rangle\otimes\left|\tfrac12,-\tfrac12\right\rangle$$

$$\doteq \left(\begin{array}{c} 0 \\ 0 \\ \hdashline 0 \\ \sqrt{2} \\ \hdashline 1 \\ 0 \end{array}\right)$$

by using the orthogonality ⊕ "convention",

we may write

$$\left|\tfrac12,\tfrac12\right\rangle = -\sqrt{\tfrac13}\,|1,0\rangle\otimes\left|\tfrac12,\tfrac12\right\rangle + \sqrt{\tfrac23}\,|1,1\rangle\otimes\left|\tfrac12,-\tfrac12\right\rangle$$

$$\left|\tfrac12,-\tfrac12\right\rangle = -\sqrt{\tfrac23}\,|1,-1\rangle\otimes\left|\tfrac12,\tfrac12\right\rangle + \sqrt{\tfrac13}\,|1,0\rangle\otimes\left|\tfrac12,-\tfrac12\right\rangle$$

⇒ "Clebsch-Gordan Coefficients" ☆☆☆

$$\begin{pmatrix} \left|\tfrac32,\tfrac12\right\rangle \\ \left|\tfrac12,+\tfrac12\right\rangle \end{pmatrix} = \underbrace{\begin{pmatrix} \sqrt{\tfrac23} & \sqrt{\tfrac13} \\ -\sqrt{\tfrac13} & \sqrt{\tfrac23} \end{pmatrix}}_{\text{orthogonal matrix.}} \begin{pmatrix} |1,0\rangle\otimes\left|\tfrac12,\tfrac12\right\rangle \\ |1,1\rangle\otimes\left|\tfrac12,-\tfrac12\right\rangle \end{pmatrix}.$$

So, we may construct a unitary transformation from $\{|l,m_l\rangle\otimes|s,m_s\rangle\}$ to $\{|j,m\rangle\}$ as

$$\mathcal{U}\left(|l,m_l\rangle\otimes|s,m_s\rangle\right) = |j,m\rangle .$$

「"$1\otimes\frac{1}{2} = \frac{3}{2}\oplus\frac{1}{2}$" : $3\times 2 = 4+2$.」

$$\Rightarrow\quad \mathcal{U} \doteq \begin{array}{cc} & \begin{array}{cccccc} m_l=1 & & m_l=0 & & m_l=-1 & \\ m_s=\frac{1}{2} & m_s=-\frac{1}{2} & & & & \end{array} \\ & \left(\begin{array}{cc:cc:cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1/3} & \sqrt{2}/\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}/\sqrt{3} & 1/\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hdashline 0 & \sqrt{2}/\sqrt{3} & -1/\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\sqrt{3} & -\sqrt{2}/\sqrt{3} & 0 \end{array}\right) \end{array} \begin{array}{l} j\ m \\ \cdots |\frac{3}{2},\frac{3}{2}\rangle \\ \cdots |\frac{3}{2},\frac{1}{2}\rangle \\ \cdots |\frac{3}{2},-\frac{1}{2}\rangle \\ \cdots |\frac{3}{2},-\frac{3}{2}\rangle \\ \cdots |\frac{1}{2},\frac{1}{2}\rangle \\ \cdots |\frac{1}{2},-\frac{1}{2}\rangle \end{array}$$

"multiplet"

「c.f.) triplet, singlet in $\frac{1}{2}\otimes\frac{1}{2}$」

Verify:

$$\mathcal{U} J_z \mathcal{U}^\dagger = \hbar \left(\begin{array}{cccc:cc} \frac{3}{2} & & & & & \\ & \frac{1}{2} & & & & \\ & & -\frac{1}{2} & & & \\ & & & -\frac{3}{2} & & \\ \hdashline & & & & \frac{1}{2} & \\ & & & & & -\frac{1}{2} \end{array}\right).$$

* How to read out the table of CG coefficients

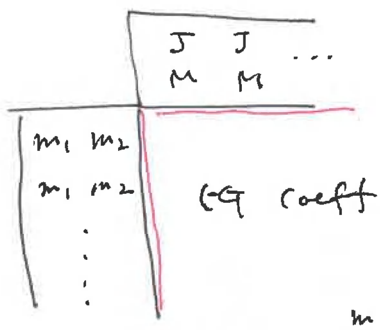
"just Google it! you'll find the table."

$1\times 1/2$		$3/2$ $+3/2$		
$+1$	$+1/2$	1	$3/2$ $+1/2$	$1/2$ $+1/2$
	$+1$	$-1/2$	$1/3$	$2/3$
	0	$+1/2$	$2/3$	$-1/3$

$|\frac{3}{2},+\frac{1}{2}\rangle$

$|1,1\rangle\otimes|\frac{1}{2},-\frac{1}{2}\rangle$

square-root is omitted: $\sqrt{2/3}$



$$\overset{j}{|}\tfrac{3}{2},\overset{m}{\tfrac{1}{2}}\rangle = \frac{1}{\sqrt{3}}\,|1,1\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\,|1,0\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

also,

$$|1,\overset{m_1}{1}\rangle\otimes|\tfrac{1}{2},\overset{m_2}{-\tfrac{1}{2}}\rangle = 1/\sqrt{3}\,|\tfrac{3}{2},\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\,|\tfrac{1}{2},\tfrac{1}{2}\rangle .$$

There're also computation codes to calculate CG coeffs. in python, C/C++, ...